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ANALYZING SPLIT TIMES FOR RUNNERS IN THE 2013 ST. GEORGE MARATHON

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Marathon participation has exploded in recent years. Participants range in ability, but the thrill of running farther and faster seems generally common. In this study, we explore marathon pacing through checkpoint splits for participants in the 2013 St. George Marathon, a field consisting of over 5000 finishers. We explore how the elite runners are pacing relative to the average, how men and women differ in pacing, and how pacing profiles change across age.

Using a Bayesian form of multivariate regression, we analyze posterior distributions for effect sizes on gender, age, and ability, as well as first order interactions, on the relative pace strategy of participants. The goal is for marathon participants and coaches to be able to use this inference to make more educated decisions on pace strategy.

1. Motivation. Most marathon runners are likely familiar with the painful experience of the last few miles of a marathon when nutrition, hydration, or proper pacing was neglected or improperly managed in early stages of the race. The marathon is difficult, because a runner's inevitable future fatigue can be masked by how good the marathon can feel in early stages.

For these reasons and others, coaches often advise their athletes to drink even if they don't feel like it, and to trust the pace even if it feels easy. Still, in early stages of the race, we participants are often tempted to venture beyond what our training indicated is the proper strategy.

In my first marathon I recall giving my wife a "thumbs-up" at half way, thinking that I felt great *and* I was going faster than I thought I could. This feeling of early triumph was squandered when at mile 21 I wondered how I was going to get to the end. Tunnel-vision and loss of form soon ensued. The finish line came with tears, of happiness and pain, but minutes later than it could have had I properly paced and fueled during the earlier stages as I had been coached.

The data show that I am not the only competitor making the mistake of careless strategy in early marathon stages. In fact, even the best of runners

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often start too fast [11]. Marathon running is currently a hot topic in modern research, but much is yet to be explored.

1.1. *The Marathon.* Interest in the marathon has grown since its inauguration in the modern Olympics. Each year hundreds of thousands of participants complete marathons across the United States. While goals for competitors differ, evidence supporting desire to improve across marathoners seems prevalent.

Displayed below in Fig. 1 is a plot indicating the estimated number of marathon finishers in the US by year.[1]

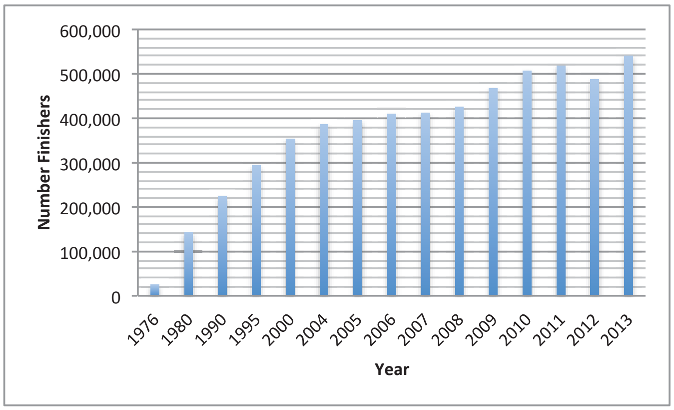

Fig. 1: Increase in US Marathon Participation Over Time

The number of competitors has grown at a fairly steady rate. 2013 featured over 1,100 marathons in the US with 92 of them reporting over 1000 finishers. Thousands more runners began, but were unable to finish. (Information taken open source from *Running USA*)

1.2. *Improvement.* Goals for competitors vary widely; some participants seek only to finish, while others attempt to qualify for the Boston Marathon or other elite races. Regardless of individual goals however, there seems to be evidence that runners want to improve; we want to run faster.

Age Group awards. Evidence of the prevalence of the desire to improve is manifested in the popularity of age group awards. Most marathons award runners in groupings of age, as well as overall, so that runners can be compared with fair subsets of the field. This allows participants to continue to compete outside of prime ages. Some races even create divisions based on weight.

[1] Note that the ING New York City Marathon was cancelled due to Hurricane Sandy in 2012 (47,000-plus likely finishers if held).

Further, methods have been developed to compare performances of individuals across all ages. One such method is age-graded performance, which adjusts time for finishers outside their prime relative to physiological limitations of age. Essentially, this allows for some comparison of who is best maximizing their potential for their respective age. Usage of this metric allows runners of varying ages to be compared.

Qualifying Standards. Some marathons (like Boston) have strict time requirements, which are based on a runners previous performance. This further illustrates the need for participants to maximize their potential. If runners want to compete in particular marathons, such as Boston, optimization of performance might be necessary.

Pace Strategy – Optimizing. For these reasons and others, many marathoners seem to be attempting to optimize their performance. Optimal performance has a number of variables, with one large factor being employing proper pace strategy. For the casual participant attempting to cross another item off of a bucket list, an optimal pace strategy can minimize pain and fatigue in late stages of the race. For the competitor seeking to qualify for the Boston Marathon, attention to pacing detail may be necessary when reaching the goal of qualifying can come down to precious minutes.

2. Literature Review. Expanding interest has motivated current scientific study of runners competing in marathons. Research on marathon nutrition is extensive. Evaluation of running mechanics and energy cost over the marathon distance are also rigorously explored. However, pace strategy seems less emphasized. Of the limited articles that exist on pace strategy, most have concluded that steady pacing yields better results than changing pace throughout the race. Even less research on differences in pacing strategies across gender, age, and ability level has been done. Results of relevant studies are summarized below, followed by methodology to be used.

2.1. *Even Pacing.* Theory on pace strategy is fairly diverse, and highly dependent on competition distance. For the marathon “even pacing,” i.e. constant pace, is generally regarded as an optimal strategy. However, other pacing theories have been applied to the marathon. Work by Abbiss and Laursen (2008) [1] summarizes many of these theories, and explains why particular pacing strategies could be appealed to. Exploration of how these pacing profiles change across age and gender is less known.

2.2. *Gender.* Santos-Lozano (2014) [11] studied the impact of gender and ability level for finishers of the NYC marathon from 2006-2011 (for 190,228 finishers). They concluded that there weren't significant differences in pacing profiles of males and females. However, March et al. (2011) [9] compared the final 9.7 k to the first 37.5 k of the Last Chance Marathon in Dublin, OH (about 400 finishers), and concluded that females better maintained pace through the final portion of the race.

Trumbee et al. (2014) [13] studied the relative pacing of men and women by comparing the final 12.2 k velocity to that of the first 30 k from the results of the Chicago Marathon in 2007 and 2009. They concluded that elite male and female runners exhibited similar pacing profiles, while average males and females differed. In these marathons, the average females were significantly closer to even pacing than their male counterparts.

In a study regarding elite triathletes, Le Meur et al. (2009) [8] concluded that men and women did differ in some aspects of their pacing profiles. Women in general seemed to control their pace better through the final leg of the iron man triathlon (or marathon), but were more affected by late hills than the men were.

2.3. *Level.* Santos-Lozano (2014) [11] also concluded that subjects of all abilities typically adopt a positive pacing profile. This positive pacing effect, or slowing over the course of the marathon, was tempered by ability. The faster runners exhibited less change, or slowed less.

In Santos-Lozano's study, the slowing of pace was calculated for groups of varying ability by finish time. The groups included: Fastest, Fast, Medium, and Slow. The coefficient of variation (or ratio of standard deviation to the mean) for these runners was calculated across groups for speed between each of the 5k split times. From Fastest to Slow, each group progressively saw a higher coefficient of variation, with that of the Fastest group about half that of the slow group.

Ely et al. (2008) [5] concluded that the elite runners exhibit even pacing across increased ambient temperature that is exponentially better than that of average runners. Trubee et al. (2014) [13] concluded similarly with his findings from the Chicago marathon. Maughan et al. (1985) [10] also noted that the elite have a more even pacing profile than average runners.

2.4. *Age.* To my knowledge, only one relevant article has been published on the effects of age on pacing. March et al. (2011) [9] found the effects of age to be significant. Results concluded that runners slow with age, pacing profile wasn't compared.

3. Bayesian Review. Bayesian statistics is a form of analysis in which a prior belief about a phenomenon can be updated with new information from the data to get a posterior belief about the phenomenon. The methodology of Bayesian statistics is based on what is now called Bayes' theorem. His work was published after his death (Bayes and Price (1763) [2]), and was extended by de Laplace (1774) [3] a few years later.

3.1. *Bayes' Theorem.* Bayes' theorem says that the probability of event A given event B has already occurred is:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)},$$

where $P(B)$ can be partitioned as:

$$P(B) = P(B|A)P(A) + P(B|A')P(A'),$$

where A' denotes the complement of A. The theory can extend to a countable set of events A_i for $i = 1, ..., n$ given B as follows:

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}.$$

where $A_i \cap A_j = \emptyset \ \forall \ i \neq j$. This can then be extended, as in Gelman et al. (2013) [6], to probability distributions $p(\theta|y)$. In this application, θ is a parameter (or set of parameters), and y are the data. Bayes' theorem then states

$$p(\theta|y) = \frac{f(y|\theta)\pi(\theta)}{\int f(y|\theta)\pi(\theta)d\theta}.$$

$f(y|\theta)$ denotes the likelihood of the data y given a set of parameters θ, and $\pi(\theta)$ is the prior distribution of θ. $p(\theta|y)$ is then the posterior distribution of θ given the data, and represents a combination of what was previously believed about θ and what the data implied.

3.2. *Reason for a Bayesian Approach.* The Bayesian approach is desirable for a variety of reasons. Foremost, the uncertainty about a parameter is summarized by the posterior distribution. In other approaches, the parameters are estimated and decisions are made based on an assumed null value of the parameters.

Another argument for this approach is that the methodology can be applied to a huge variety of problems. With a likelihood and a prior distribution that conform to some regularity conditions standard for probability density functions (that can even be relaxed in some cases), and some estimation techniques for estimating integrals with no easy closed form solution, posterior distributions for θ can be estimated. This allows for problems where maximum likelihood approaches are not easily solvable to be answered with straightforward methodology.

Some argue that the Bayesian approach isn't objective enough for science because the posterior distribution is influenced by a subjective prior. In this analysis, a locally uniform diffuse (improper) prior will be used, allowing the data to entirely drive the posterior distributions on the parameters.

Additional Bayesian theorems outlined below further justify the straightforward methodology of Bayesian statistics and its general application.

3.3. *Conjugate Prior.* If the prior distribution has the same functional form as the likelihood, then the prior is called conjugate (Gelman et al. (2013)[6], pg. 35-36). If a prior is conjugate for the likelihood, than the solution to $p(\theta|y)$ previously discussed is a closed form distribution of the form of the prior, and can be solved analytically. See Diaconis et al. (1979) [4] for a more complete presentation.

3.4. *Gibbs Sampler.* For cases that aren't conjugate, Marcov chain Monte Carlo (MCMC) techniques can be used to estimate posterior distributions of parameters by getting draws from that posterior distribution.

Named for work of the physicist Josiah Willard Gibbs, Geman and Geman (1984) [7] outline the process of Gibbs sampling. Gibbs sampling is a special case of MCMC in which the posterior distribution of a parameter (θ_i) can be found given the values of the other parameters (θ_{-i}) in the likelihood and prior distribution for that parameter (θ_i). In other words, if the distributions

$$[\theta_1|\theta_2, \theta_2, ..., \theta_r, y]$$

$$[\theta_2|\theta_1, \theta_3, ..., \theta_r, y]$$

$$\vdots$$

$$[\theta_r|\theta_1, \theta_2, ..., \theta_{r-1}, y]$$

can be solved for in closed form, Gibbs sampling shows that with reasonable starting values, an iterative process of updating each parameter one at a time will converge to the posterior distributions of the parameters.

3.5. *Bayesian Regression.* An interesting application of the Gibbs sampler is linear regression. Linear regression takes the form

$$y_i = \beta_0 + \beta_1 X_{i1} + ... + \beta_q X_{iq} + \epsilon_i,$$

for observations $i = 1, ..., n$. β_k for $k = 1, ..., q$ is then the estimated effect size for corresponding x_{ik} on y_i. If we assume $\epsilon_i \overset{iid}{\sim} N(0, \sigma^2)$ then

$$\underline{y} \sim N(X\underline{\beta}, \sigma^2).$$

Conjugate priors for β and σ, are normal and inverse gamma respectively. That is, if

$$\pi(\beta) \sim N(m, S)$$
$$\pi(\sigma^2) \sim IG(a, b)$$

then

$$[\beta] \sim N\Big(\Big(\frac{X'X}{\sigma^2} + \frac{1}{S}\Big)^{-1}\Big(\frac{X'y}{\sigma^2} + \frac{m}{S}\Big), \Big(\frac{X'X}{\sigma^2} + \frac{1}{S}\Big)^{-1}\Big)$$
$$[\sigma^2] \sim IG\Big(\frac{n}{2} + a, \frac{\sum_{i=1}^{n}(y_i - x_i\beta)^2}{2} + b\Big)$$

This form generalizes to the multivariate case, and is outlined by Tiao and Zellner (1964) [12]. Formally

$$y_{ij} = \beta_{0j} + \beta_{1j}X_{i1} + ... + \beta_{qj}X_{iq} + \epsilon_{ij},$$

which is similar to the univariate case, but additionally for $j = 1, ..., p$, the number of columns in the y matrix. If we assume constant priors (as done by Tiao and Zellner) then $\pi(\beta)$ and $\pi(\Sigma)$ take the form

$$\pi(\beta) = M$$

$$\pi(\Sigma) \propto |\Sigma|^{-\frac{1}{2}(m+1)}$$

The joint posterior of this multivariate case then is

$$p(\beta, \Sigma|y) \propto p(\beta|\Sigma, y)p(\Sigma|y)$$

where Σ is the variance covariance matrix for y and has dimension p by p. Tiao and Zellner further outline that posterior distributions for β and Σ follow multivariate and inverse-Wishart distributions respectively.

4. Application. An application of this Bayesian multivariate normal regression will be used in this marathon analysis. There are a number of interesting covariates for runners participating in the marathon that affect pace. Times for multiple checkpoints recorded throughout the marathon will serve as the random variables to be estimated. Posterior distributions on β and Σ as outlined provide a great way to analyze effect sizes for these covariates on pacing.

4.1. *St. George Marathon Data.* We have data from the 2013 St. George Marathon. The field consisted of over 5000 finishers. Basic summary statistics are included in Table 1. *BQ* refers to total number of Boston Marathon Qualifiers.

	Gender		Age		Time
M:	3166	Min	8	Fastest	2:15:56
F:	2653	1st Quartile	33	1st Quartile	3:38:15
		Median	40	Median	4:07:41
BQ:	824	3rd Quartile	49	3rd Quartile	4:47:42
		Max	81	Slowest	7:21:49

Table 1
Distribution of Finishers for 2013 St. George Marathon

We have pace between splits for 3 mid-way checkpoints and the finish. The response variable for each participant is pace for 4 midway splits[2] divided by that participant's average pace for the marathon. This metric allows us to compare how sub-groups of finishers are pacing the marathon relative to each runner's pace. This analysis isn't intended to determine how much faster the elite are running relative to the field, but rather to determine how the elite are pacing. A runner finishing in 4 hours with even pacing across splits will have the same response as a runner finishing in 3 hours with an even pace. Summary statistics on response are included in Table 2.

	pace1	pace2	pace3	pace4
Min.	0.8843	0.7276	0.7184	0.6829
1st Qu.	1.0187	0.9543	1.0105	0.9229
Median	1.0563	0.9720	1.0315	0.9653
Mean	1.0741	0.9774	1.0270	0.9580
3rd Qu.	1.1124	0.9953	1.0484	0.9986
Max.	1.9037	1.2580	1.4835	1.8372

Table 2
Summary of the Response

The following box plot (Fig. 2) illustrates the distribution of the response. For the St. George marathon specifically, the response makes intuitive sense. The topography for the marathon is downhill for splits 1,3, and 4–with the majority of the decline in split 3–and flat for split 2. So given that marathon runners typically start at a faster rate than they finish, yet run faster on

[2]start-10k; 10k-21k; 21k-30k; 30k-finish

downhill portions (specifically split 3), this response agrees with what we would expect.

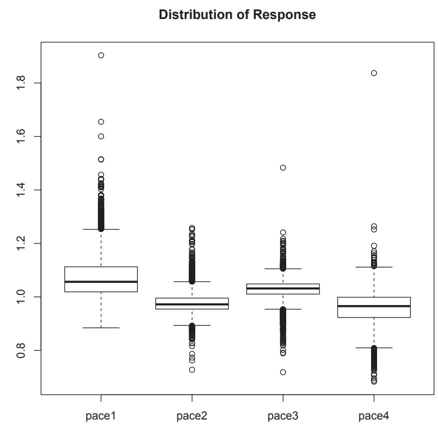

Fig 2: Distribution of Response Variables

4.2. *Model.* The proposed model is then

$$y_i \sim MVN(X_i\beta, \Sigma),$$

where y is n by p for $n = 5819$ participants, and $p = 4$ adjusted split paces. X will be of dimension n by q. X_{i1}, is an indicator for male, X_{i2} is the centered and scaled age, X_{i3} is an indicator for hitting the Boston qualifier, and X_{i4} is an indicator for elite (top 50 males and females). We then have a second order polynomial term for age, and first order interactions for gender, age, and Boston qualifier. The parameters of this model correspond to effect sizes of linear combinations of partitions of the data, and the distributions of these parameters will offer insight into effects for age, gender, ability level, etc.

Using Gibbs sampling, β and Σ are updated iteratively where the posterior distribution on the β's are obtained through methods outlined by Tiao and Zellner [12]. β takes on the following form

$$\beta|\Sigma \sim MVN(\hat{\boldsymbol{\beta}}, \Sigma \otimes (X'X)^{-1})$$

where

$$\Sigma \sim W^{-1}(n, S)$$

$$S = (y - X\beta)'(y - X\beta) + (\beta - \hat{\beta})'X'X(\beta - \hat{\beta}).$$

$\hat{\beta} = (X'X)^{-1}X'y$, and has dimension 9 by 4. $\Sigma \otimes (X'X)^{-1}$ is a 36 by 36 square positive definite covariance matrix. Further, W^{-1} is the inverse Wishart Distribution, and n=5766.

Priors implicit are non-informative. The decision to model with uninformative priors was debated, but with confounding results on effects of covariates on pace recorded in previous studies, priors were left as uninformative so as to not drive the results.

We believe this model will sufficiently describe the effects of these covariates on pace for the St. George marathon. This will allow for more complete inference than what has been done in previous studies.

The assumptions include normal distribution of errors. This seems reasonable in this case, but will be addressed in more detail later. Additionally, we assume an additive model and make other distributional assumptions explicit on β and Σ.

5. Results. The results verify some of the conclusions summarized in the literature review previously, in addition to contributing new theory on marathon pacing profiles for runners of the St. George Marathon.

5.1. *Posterior Distributions.* Gibbs sampling with a burn in of 100 was used to obtain 10,000 draws that estimate posterior distributions on β and Σ, and a summary on posterior distributions follow.

	Split 1	Split 2	Split 3	Split 4
$Intercept$	1.0696	0.9703	1.0254	0.9666
$Male$	0.0089	0.0174	0.0054	-0.0216
Age	-0.0048	-0.0034	0.0030	0.0028
$Boston$	-0.0449	-0.0048	0.0132	0.0217
$Top100$	-0.0048	-0.0036	0.0001	0.0055
$Age * Male$	0.0045	-0.0033	-0.0033	0.0016
$Age * Boston$	0.0018	0.0053	-0.0017	-0.0044
$Male * Boston$	-0.0140	-0.0141	-0.0031	0.0203
Age^2	0.0071	-0.0000	-0.0026	-0.0021

TABLE 3
Posterior Mean of Betas

Table 3 gives the average of the posterior distribution for each of the βs. The box plots (Fig. 3) and table of standard deviations (Table 4) for each posterior distribution that follow help to quantify uncertainty of these point estimates.

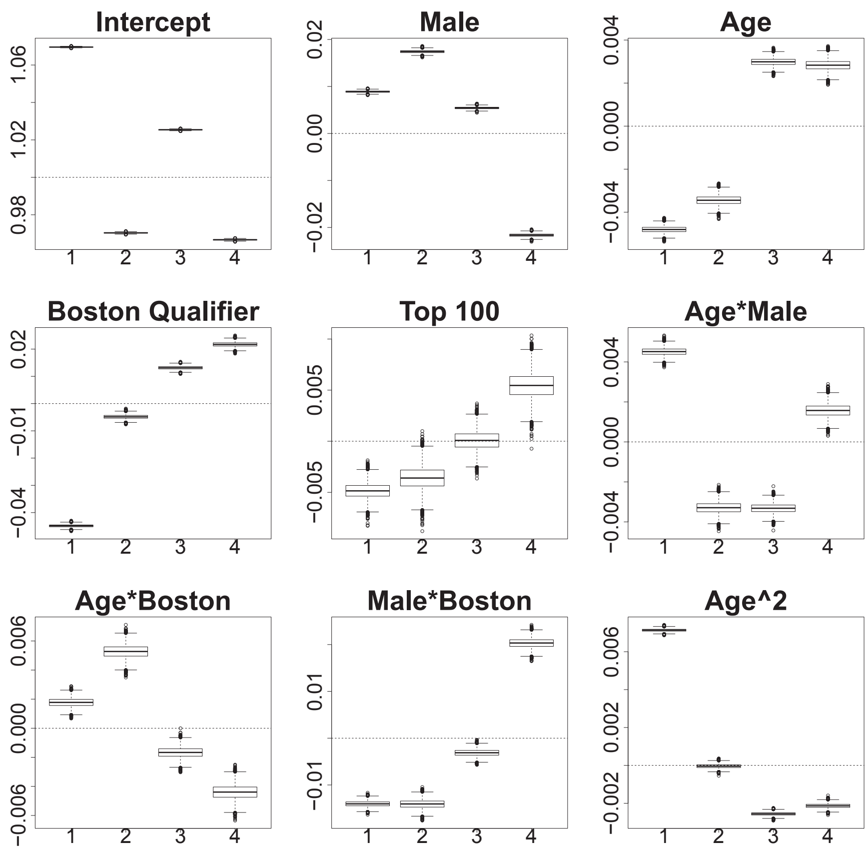

Fig 3: Boxplots of the posterior distributions of β

	Split 1	Split 2	Split 3	Split 4
$Intercept$	0.0002	0.0003	0.0002	0.0003
$Male$	0.0002	0.0003	0.0003	0.0004
Age	0.0002	0.0002	0.0002	0.0003
$Boston$	0.0003	0.0005	0.0004	0.0006
$Top100$	0.0008	0.0011	0.0009	0.0013
$Age * Male$	0.0002	0.0003	0.0002	0.0003
$Age * Boston$	0.0003	0.0004	0.0003	0.0004
$Male * Boston$	0.0005	0.0007	0.0006	0.0008
Age^2	0.0001	0.0001	0.0001	0.0001

TABLE 4
Posterior SD of Betas

Intuition tells us that the intercept for the four splits ought to average around one, and that the effect sizes for each covariate ought to average around zero. The box plots pictured previously in Fig. 3 illustrate this phenomenon.

5.1.1. *Gender.* Females are implicit in the intercept, and it is clear that males are running differently than the females, but that the elite male and females are running similarly.

5.1.2. *Boston Qualifiers.* The intercept represents an average 40 year old female that did not achieve a Boston Marathon qualifying mark. Relative to her qualifying counterpart, the average 40 year old female started with a faster pace, and slowed at the end. The effect for Boston qualifier is interesting. Although marginal inference of this effect in the presence of significant interactions needs to be made less generally, the Boston Marathon qualifiers seem to be starting the race more conservatively, and finishing faster. This validates conclusions about the more elite runners general to research previously discussed ([11], [5], [13], [10]).

Note additionally that the effect for $Top100$ does not have near the impact on pacing as the effect for $Boston\ Qualifier$. This seems to illustrate that those qualifying for the Boston Marathon are capturing this *elite* effect. Those top finishers of the St. George Marathon are not pacing much differently than the hundreds of Boston Qualifiers that follow them, but the group of Boston Qualifiers are significantly different in pacing profile than the thousands of other finishers that follow.

Comparisons on splits for 30 and 50 year old runners, profiled by gender and ability are presented in tables 5 and 6 that follow (emphasis added for easier comparison). Note that the Boston Qualifiers (BQ) are generally starting more conservatively and finishing stronger relative to their non-qualifying counterparts.

Gender	BQ	Split 1	Split 2	Split 3	Split 4
Male	**Yes**	**1.02**	0.97	1.04	**0.98**
Male	No	1.08	0.99	1.03	0.94
Female	**Yes**	**1.03**	0.96	1.04	**0.99**
Female	No	1.09	0.97	1.02	0.96

TABLE 5
Expected Splits for Runners ***Age 30*** *(emphasis added for comparison)*

Gender	BQ	Split 1	Split 2	Split 3	Split 4
Male	**Yes**	**1.02**	0.96	1.04	**0.99**
Male	No	1.08	0.98	1.03	0.95
Female	**Yes**	**1.03**	0.97	1.04	**0.99**
Female	No	1.08	0.97	1.02	0.96

TABLE 6
Expected Splits for Runners ***Age 50*** *(emphasis added for comparison)*

5.1.3. *Age.* The posterior distributions on the age effect imply that on average as runners age, they start the race more conservative, and finish faster relative to their younger counterparts. So as runners age, their pacing profiles closer resemble that of the faster runners. However, this effect is tempered by extreme ages, as the effect sizes on Age^2 illustrate. On average, as runners get increasingly old, they revert to starting more aggressively than they are finishing.

This effect is illustrated in Tables 7 and 8 (emphasis added for easier comparison). Note that estimates for the comparisons that follow are made using the mean of the posterior distributions on β. Confidence bands for split estimates are not included because presentation of estimates for very many cases would make tables overly complex. Note that variances for posterior distributions of β (included in Table 4) are relatively small.

As runners age, on average they are starting more conservatively, an effect we noted common to the more elite runners. Yet as those ages creep beyond 50's, runners seem to revert back to tendencies common to younger runners. Perhaps this illustrates that as participants age, they want to continue to perform at a level they once could, but their bodies just can't handle the pace that they could at younger ages.

Age	Split 1	Split 2	Split 3	Split 4
10	1.13	1.00	1.01	0.92
20	1.10	1.00	1.02	0.93
30	**1.08**	0.99	1.03	0.94
40	**1.08**	0.99	1.03	0.94
50	**1.08**	0.98	1.03	0.95
60	1.10	0.98	1.02	0.95
70	1.12	0.97	1.01	0.94
80	1.16	0.96	1.00	0.94

Table 7

Effect of Age on ***Males*** *(non-Boston qualifying)*

Age	Split 1	Split 2	Split 3	Split 4
10	1.13	0.98	1.00	0.94
20	1.10	0.98	1.01	0.95
30	**1.09**	0.97	1.02	0.96
40	**1.08**	0.97	1.02	0.96
50	**1.08**	0.97	1.02	0.96
60	1.10	0.96	1.02	0.96
70	1.12	0.96	1.01	0.95
80	1.16	0.96	1.00	0.94

Table 8

Effect of Age on ***Females*** *(non-Boston qualifying)*

5.1.4. *Correlations.* Correlations on the posterior distributions of the β's also illustrate some interesting phenomena. Refer to the following correlation plot (Fig. 4).

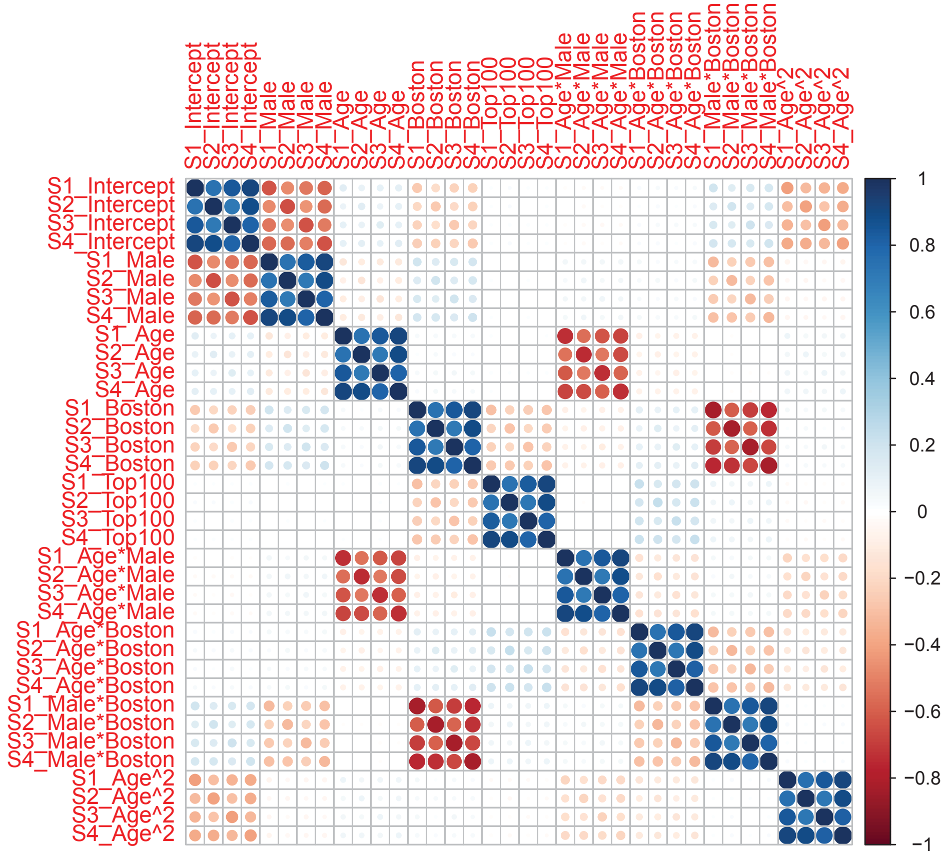

Fig.4: Correlations of the Posterior Distributions of β

Correlations between covariates and their first-order interactions are present in many cases. Colinearities should be noted, but are not surprising. Some interesting correlations are those that correlate to the intercept. For example, the strong negative correlation between males and the intercept suggest that males and females behave inversely in their pacing of the marathon.

5.1.5. *Posterior Distribution of* Σ. The posterior for Σ is a bit harder to present. The element-wise mean of the posterior distribution is pictured in Table 9, and corresponding correlation matrix in Table 10.

	Split 1	Split 2	Split 3	Split 4
Split 1	0.00005	0.00005	0.00005	0.00007
Split 2	0.00005	0.00011	0.00006	0.00011
Split 3	0.00005	0.00006	0.00007	0.00008
Split 4	0.00007	0.00011	0.00008	0.00014

Table 9
Posterior Mean for Σ (element-wise)

	Split 1	Split 2	Split 3	Split 4
Split 1	1.00	0.74	0.73	0.94
Split 2	0.74	1.00	0.58	0.92
Split 3	0.73	0.58	1.00	0.71
Split 4	0.94	0.92	0.71	1.00

Table 10
Posterior Correlation Matrix for Σ

The correlations are interesting. What a runner runs for splits 1 and 2 is more highly correlated with how they will finish than the more downhill split 3. This would imply that downhill portions affect runners differently. Implications would include that we need to be careful in concluding results found in this marathon data apply to other marathons of differing topography.

5.2. *Model Fit Diagnostics.* The fit seems reasonable, however there are some assumptions of this model that deserve attention. One assumption is normally distributed error.

The first split exhibits more of a right skew with more runners going out much faster than they can maintain, and few runners starting much under pace. The last split also has some skewness in the other direction, with some runners finishing much slower than their average, while few runners finish much faster than their average. This can be seen by a careful examination of Table 2, and the box plots featured in Fig. 2.

This model misspecification doesn't seem to have drastic effects on the results. Using posterior predictive checking, the model seems to have reasonable fit. The quantiles of each held-out split relative to the remainder used to fit the model look fairly uniform. And in each of the 4 cases, the Kolmogorov-Smirnov test for uniformity fails to reject.[3]

Seeing the model fit seems reasonable, we like this model over competing models because the interpretability of the multivariate normal model is clear,

[3]see Appendix A for a more detailed explanation of posterior predictive checking as well as histograms for the quantiles.

as direct inference can be made on posterior distributions. Also the ease of Gibbs sampling in this model is preferred.

6. Conclusions. These data confirm previous conclusions on the pacing nature of the more elite runners ([11], [5], [13], [10]). The elite do a better job at maintaining their starting pace, slowing less throughout the race. Perhaps the faster runners are just better at guessing the right pace, but more likely these runners are privy to superior coaching, and better strategy.

An interesting corollary is that runners pacing profile seem to closer emulate that of the elite as they age. This could be evidence that marathoners are learning preferred pacing strategy through experience. Distance runners hit late primes, often into their upper 30s. Perhaps runners even past prime ages can continue to improve, countering physical limitations of age with time-tested race strategy.

Previous studies regarding the effect of gender in pacing profiles of the marathon were confounding. In these data, we see the males starting more conservatively (an effect we see with the Boston Qualifiers) relative to the females, but finishing slower (an effect opposite to that of the qualifiers) than the females.

Advise to marathoners seeking to improve performance would be to pattern pacing after what the elite and experienced runners are doing–start the race more conservative, and run the downhill portions faster. Additionally, as runners age beyond their 50s, we caution them to be realistic in setting time goals; we advise these athletes to start the race more conservatively.

We hope the results can be useful for coaches and marathon athletes in general as each strive for more optimal pacing.

APPENDIX A: POSTERIOR PREDICTIVE

Posterior predictive checking is treated in detail by Gelman et al. (2013) [7], but in short is based on theory regarding the probability integral transform. Because the distribution of the inverse of a valid CDF is uniform, a plot of the quantiles of estimates should be uniform.

The test is done by removing $y_{i.}$, then fitting the model to get the posterior distributions of the parameters. All $y_{-i.}$ are estimated using the mean of the posterior distribution on each β, then $y_{i.}$ are estimated using the same β, and the percentile of where each estimation of $y_{i.}$ fall relative the estimates of $y_{-i.}$ is recorded. This is done for all $y_{i.}$, $i = 1, ..., 5766$. These percentiles (for each $y_{.j}$, where $j = 1, ..., 4$) should follow a uniform distribution. The Kolmogorov-Smirnov test for uniformity was used to test if each of the 4 distributions significantly differed from the uniform(0,1), which they did not at the $\alpha = .05$ level. The histograms for each posterior prediction follow.

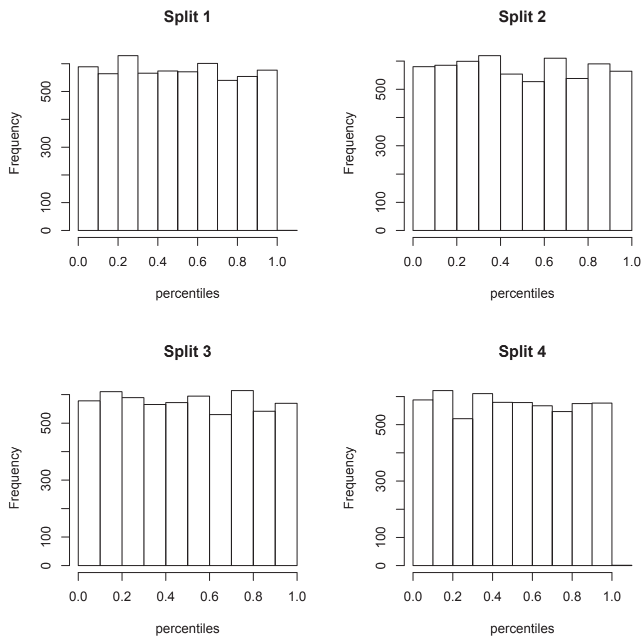

Fig. A: Histograms on the Percentiles for each of the 4 Posterior Predictive Distributions

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REFERENCES

[1] Abbiss, C. R. and Laursen, P. B. (2008). Describing and understanding pacing strategies during athletic competition. *Sports Medicine* **38** 239–252.

[2] Bayes, M. and Price, M. (1763). An Essay towards solving a Problem in the Doctrine of Chances. By the late Rev. Mr. Bayes, FRS communicated by Mr. Price, in a letter to John Canton, AMFRS. *Philosophical Transactions (1683-1775)* 370–418.

[3] de Laplace, P. (1774). Mémoire sur les suites récurro-récurrentes et sur leurs usages dans la théorie des hasards. *Mém. Acad. Roy. Sci. Paris* **6** 353–371.

[4] Diaconis, P., Ylvisaker, D. et al. (1979). Conjugate priors for exponential families. *The Annals of statistics* **7** 269–281.

[5] Ely, M. R., Martin, D. E., Cheuvront, S. N. and Montain, S. J. (2008). Effect of ambient temperature on marathon pacing is dependent on runner ability. *Medicine and science in sports and exercise* **40** 1675–1680.

[6] Gelman, A., Carlin, J. B., Stern, H. S., Dunson, D. B., Vehtari, A. and Rubin, D. B. (2013). *Bayesian data analysis*. CRC press.

[7] Geman, S. and Geman, D. (1984). Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images. *Pattern Analysis and Machine Intelligence, IEEE Transactions on* **6** 721–741.

[8] Le Meur, Y., Hausswirth, C., Dorel, S., Bignet, F., Brisswalter, J. and Bernard, T. (2009). Influence of gender on pacing adopted by elite triathletes during a competition. *European journal of applied physiology* **106** 535–545.

[9] March, D. S., Vanderburgh, P. M., Titlebaum, P. J. and Hoops, M. L. (2011). Age, sex, and finish time as determinants of pacing in the marathon. *The Journal of Strength & Conditioning Research* **25** 386–391.

[10] Maughan, R., Leiper, J. and Thompson, J. (1985). Rectal temperature after marathon running. *British journal of sports medicine* **19** 192–195.

[11] Santos-Lozano, A. (2014). Influence of Sex and Level on Marathon Pacing Strategy. Insights from the New York City Race. *International Journal of Sports Medicine*.

[12] Tiao, G. C. and Zellner, A. (1964). On the Bayesian estimation of multivariate regression. *Journal of the Royal Statistical Society. Series B (Methodological)* 277–285.

[13] Trubee, N. W., Vanderburgh, P. M., Diestelkamp, W. S. and Jackson, K. J. (2014). Effects of Heat Stress and Sex on Pacing in Marathon Runners. *The Journal of Strength & Conditioning Research* **28** 1673–1678.

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